

Vertical epicyclic stability and dissipative settling in a weakly deformed q-metric spacetime

A. Muratkhan¹ , A.S. Taukenova¹ , A.A. Dalelkhankyzy² , A.T. Orazymbet^{*1} 

¹Department of Theoretical and Nuclear Physics, Al-Farabi Kazakh National University, Almaty, Kazakhstan

²Department of Physics, Kazakh National Women's Teacher Training University, Almaty, Kazakhstan

E-mail: muratkhan.array@kaznu.kz, liya_tauken@mail.ru, dalelkhankyzy.a@qyzpu.edu.kz,
*ayazhan.orazymbet@kaznu.edu.kz

Abstract. We investigate the motion of massive test particles close to the equatorial plane of a weakly deformed q-metric spacetime. Working to first order in the dimensionless quadrupole parameter q , we derive the orbital angular velocity and the vertical epicyclic frequency for equatorial circular orbits while keeping the physical monopole mass fixed. The Schwarzschild limit is recovered at $q = 0$, whereas a nonzero quadrupole removes the degeneracy between the vertical and azimuthal frequencies already at $O(q)$.

A small phenomenological damping term is then introduced into the linearized vertical equation to describe the gradual approach of a perturbed orbit toward the equatorial plane. In the conservative case, the particle continues to oscillate about this plane without any decrease in amplitude. With damping, the amplitude decays exponentially. The quadrupolar deformation changes the vertical oscillation frequency and, consequently, the number of cycles completed before a specified settling threshold is reached. It does not, however, set the decay timescale, which follows from the adopted damping prescription. Since the coordinate-time damping coefficient is kept fixed in our numerical examples, the trajectories share the same exponential envelope but oscillate at different frequencies.

The comparison with the exact q-metric also provides a direct check of the first-order approximation. Over the sampled domain $|q| \leq 0.1$ and $6 \leq r/M \leq 20$, the relative error in the vertical frequency remains below 0.34%, while the corresponding frequency ratio is reproduced to better than 0.10%. These results show explicitly how the quadrupolar geometry modifies vertical orbital dynamics within a model in which dissipative settling is introduced independently.

Keywords: q-metric; quadrupole deformation; circular geodesics; vertical epicyclic frequency; dissipative settling.

Received 30.07.2026. Revised 21.08.2026. Accepted 31.08.2026. Available online 30.09.2026

*the corresponding author

Introduction

The Schwarzschild solution provides the standard reference geometry for a static spherical source. Owing to spherical symmetry, all orbital planes are equivalent, and the vertical epicyclic frequency of a circular orbit coincides with its azimuthal frequency. A static quadrupolar deformation breaks this degeneracy without introducing frame dragging. It therefore provides a useful setting in which the effects of nonsphericity can be separated from those associated with rotation.

The q -metric, also known as the Zipoy–Voorhees or γ -metric, is an exact static and axisymmetric vacuum solution [1, 2]. In the convention adopted here, $q=0$ gives the Schwarzschild spacetime, whereas $q \neq 0$ introduces a mass quadrupole together with a modified higher-multipole structure [3, 4]. Its singularity structure, geodesic motion, and circular-orbit properties have been studied in several contexts [5–9].

Epicyclic frequencies characterize the response of a circular geodesic to small radial or vertical perturbations. They also enter particle-based models of oscillations in accretion discs and slender tori [10–17]. Beyond the test-particle description, deformations of the central source can modify both the equilibrium structure of surrounding relativistic fluids and the properties of thin accretion discs [18, 19]. More general relations between the spacetime geometry, its multipole moments, and the epicyclic frequencies are known for broad classes of axisymmetric spacetimes [20–22]. Geodesic observables have also been expressed directly in terms of multipole moments in extended theories of gravity [23].

Exact orbital and epicyclic frequencies in the γ -metric, together with their expansions around the Schwarzschild limit, have already been obtained in Ref. [24]. We therefore do not claim a new derivation of the complete exact-frequency structure. The contribution of the present work is instead fourfold. First, the linear corrections are reformulated at fixed physical monopole mass M , rather than at fixed metric parameter m , thereby separating the quadrupolar deformation from a simultaneous change in the monopole scale. Second, the ratio Ω_θ/Ω is used as a dimensionless and scale-independent measure of the quadrupole-induced splitting between the vertical and azimuthal frequencies. Third, the $O(q)$ expressions are compared directly with the exact q -metric results, allowing the accuracy domain of the linear approximation to be quantified for $|q| \leq 0.1$ and $6 \leq r/M \leq 20$. Finally, the resulting vertical frequency is incorporated into a minimal weak-damping model that separates the geometrical modification of the oscillation phase from the dissipative timescale governing the decay of the amplitude.

We assume $|q| \ll 1$ and retain terms through first order in q . The line element is

$$\begin{aligned}
 ds^2 = & \left(1 - \frac{2m}{r}\right) \left[1 + q \ln\left(1 - \frac{2m}{r}\right)\right] dt^2 \\
 & - r^2 \left[1 - q \ln\left(1 - \frac{2m}{r}\right)\right] \sin^2\theta d\phi^2 \\
 & - \left[1 + q \ln\left(1 - \frac{2m}{r}\right) - 2q \ln\left(1 - \frac{2m}{r} + \frac{m^2}{r^2} \sin^2\theta\right)\right] \\
 & \times \left[\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2\right] + O(q^2)
 \end{aligned} \tag{1}$$

We use the signature $(+ - - -)$ and geometrized units $G=c=1$. Setting $q=0$ removes all deformation terms and recovers the Schwarzschild line element in standard spherical coordinates. Throughout the paper, physical shifts are measured relative to this spherical baseline. The exact q -metric is employed only to assess the truncation error of the $O(q)$ approximation.

A stable vertical perturbation does not by itself produce irreversible settling. In a conservative spacetime, a particle displaced from the equatorial plane continues to oscillate about it. A gradual decrease in inclination requires an additional interaction, such as drag from surrounding matter or radiation. To keep this dissipative input separate from the geometric restoring force, we introduce a weak linear damping term rather than specifying a microscopic mechanism. The q -metric then determines the vertical oscillation frequency, while the adopted damping coefficient sets the decay timescale. More detailed covariant descriptions of radiation forces and drag are available [25, 26], but require additional assumptions about the surrounding medium, radiation field, and relative motion that are outside the scope of the present local analysis.

Research Methodology

Weakly deformed q-metric

First-order metric and multipole moments

We use the short notation

$$f(r) = 1 - \frac{2m}{r}, H(r, \theta) = f(r) + \frac{m^2}{r^2} \sin^2 \theta \quad (2)$$

To first order in q , the nonzero metric components are

$$g_{tt} = f(1 + q \ln f) \quad (3)$$

$$g_{rr} = -\frac{1}{f}(1 + q \ln f - 2q \ln H) \quad (4)$$

$$g_{\theta\theta} = -r^2(1 + q \ln f - 2q \ln H) \quad (5)$$

$$g_{\varphi\varphi} = -r^2 \sin^2 \theta (1 - q \ln f) \quad (6)$$

Every product is understood consistently through $O(q)$. Thus, after an intermediate calculation, all terms proportional to q^2 or higher must be discarded.

For the standard q -metric convention used in Refs. [3, 4], the first two nonzero mass moments are:

$$M_0 = m(1 + q), M_2 = -\frac{m^3}{3} q(1 + q)(2 + q) \quad (7)$$

Their linearized forms are:

$$M \equiv M_0 = m(1 + q) + O(q^2), M_2 = -\frac{2}{3} m^3 q + O(q^2) \quad (8)$$

Thus q controls the leading departure from spherical symmetry, although it also changes the monopole mass if the metric parameter m is held fixed.

Some papers use $\delta=1+q$ or an opposite sign for the deformation parameter. We keep the convention in Eq. (7) throughout.

The approximation is not uniform arbitrarily close to $r=2m$. Although q may be small, terms such as $q \ln f$ grow when $f \rightarrow 0$. A practical condition is therefore:

$$|q \ln f(r)| \ll 1, |q \ln H(r, \theta)| \ll 1 \quad (9)$$

All plots and numerical examples must be restricted to a radial interval where these inequalities are satisfied. This limitation is more informative than the statement $|q| \ll 1$ alone.

Schwarzschild limit and physical mass

Setting in Eqs. (3)–(6) gives

$$ds^2 = f dt^2 - \frac{dr^2}{f} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \quad (10)$$

which is the Schwarzschild line element with mass parameter m . This check is also useful for finding missing angular factors or inconsistent signs.

There are two possible comparisons. The first keeps m fixed and varies q . It is algebraically simple and will be used for the derivations below. In that comparison, however, the physical mass $M=m(1+q)$ changes along with the quadrupole. The second comparison keeps M fixed. Then:

$$m = \frac{M}{1+q} = M(1 - q) + O(q^2) \quad (11)$$

Substituting Eq. (11) into a result obtained at fixed m separates the deformation from a simple change of the monopole scale. Numerical comparisons will therefore be presented in terms of r/M and $M\Omega_r$.

Equatorial circular geodesics

Conserved quantities

The particle Lagrangian is:

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \quad (12)$$

where a dot denotes differentiation with respect to proper time τ . For a massive particle, $2\mathcal{L}=1$. Stationarity and axial symmetry give two constants of motion,

$$E = g_{tt}\dot{t}, L = -g_{\varphi\varphi}\dot{\varphi} \quad (13)$$

Here E and L are the specific energy and specific azimuthal angular momentum. Consequently,

$$\dot{t} = \frac{E}{g_{tt}}, \dot{\varphi} = -\frac{L}{g_{\varphi\varphi}} \quad (14)$$

On the equatorial plane,

$$\theta = \frac{\pi}{2}, H_{eq}(r) = \left(1 - \frac{m}{r}\right)^2 \quad (15)$$

Reflection symmetry ensures that an orbit initially tangent to this plane remains in it. For a circular orbit at $r=r_\phi$, we additionally impose $\dot{r}=r''=0$.

On the equatorial plane we define the positive functions:

$$A(r) = g_{tt}\left(r, \frac{\pi}{2}\right), C(r) = -g_{\varphi\varphi}\left(r, \frac{\pi}{2}\right) \quad (16)$$

Then the normalization condition becomes

$$At^2 - C\varphi^2 = 1 \quad (17)$$

Angular velocity

The radial geodesic equation for a circular orbit gives

$$A_{,r}t^2 - C_{,r}\varphi^2 = 0 \quad (18)$$

The angular velocity measured with respect to the coordinate time is therefore

$$\Omega^2 \equiv \left(\frac{d\varphi}{dt}\right)^2 = \frac{A_{,r}}{C_{,r}} = -\frac{g_{tt,r}}{g_{\varphi\varphi,r}} \Big|_{\theta=\pi/2} \quad (19)$$

The four-velocity and the constants of motion can be written without first expanding in :

$$u^t = \frac{1}{\sqrt{A-C\Omega^2}} \quad (20)$$

$$E = \frac{A}{\sqrt{A-C\Omega^2}} \quad (21)$$

$$L = \frac{C\Omega}{\sqrt{A-C\Omega^2}} \quad (22)$$

These expressions also state the timelike-existence condition

$$A - C\Omega^2 > 0 \quad (23)$$

First-order quadrupole corrections

For the metric in Eq. (1),

$$A = f(1 + q \ln f), C = r^2(1 - q \ln f) \quad (24)$$

Substitution into Eq. (19), followed by a consistent first-order expansion, gives

$$\Omega^2 = \frac{m}{r^3} \left\{ 1 + q \left[1 + 2 \ln f + \frac{m}{r-2m} \right] \right\} + O(q^2) \quad (25)$$

Equivalently,

$$\Omega = \sqrt{\frac{m}{r^3}} \left\{ 1 + \frac{q}{2} \left[1 + 2 \ln f + \frac{m}{r-2m} \right] \right\} + O(q^2) \quad (26)$$

The correction contains logarithmic and rational terms. Far from the source it vanishes only after the physical mass normalization is handled consistently. Indeed, at fixed physical mass one must use Eq. (11) before interpreting the large- r limit.

The energy and angular momentum can be expanded just as directly. With $u=m/r$, one obtains

$$E = \frac{1-2u}{\sqrt{1-3u}} \left\{ 1 + \frac{q}{2} \left[\ln(1-2u) + \frac{u(1-u)}{(1-2u)(1-3u)} \right] \right\} + O(q^2) \quad (27)$$

$$L = \frac{\sqrt{mr}}{\sqrt{1-3u}} \left\{ 1 + \frac{q}{2} \left[\frac{1-u}{1-3u} - \ln(1-2u) \right] \right\} + O(q^2) \quad (28)$$

Both expressions reduce to their Schwarzschild values when $q=0$. They are also consistent with the timelike condition in Eq. (23); the circular orbit becomes null when its denominator approaches zero.

Equations (25)–(28) were written at fixed m . For the comparison at fixed physical mass M , Eq. (11) gives, to the same order,

$$\Omega^2 = \frac{M}{r^3} \left\{ 1 + q \left[2 \ln F + \frac{M}{r-2M} \right] \right\} + O(q^2), F = 1 - \frac{2M}{r} \quad (29)$$

The constant term present in Eq. (25) disappears. This is the expected result: at large distance the leading Keplerian frequency is governed by the same physical monopole mass for all curves.

Vertical epicyclic motion

Linearized vertical equation

Consider a small displacement from an equatorial circular orbit,

$$r = r_0, \theta = \frac{\pi}{2} + \eta(\tau), |\eta| \ll 1 \quad (30)$$

At linear order we neglect coupling to a radial perturbation. This is the standard vertical epicyclic approximation; it describes the initial response of a nearly equatorial orbit, not an arbitrary large-inclination trajectory.

The θ Euler-Lagrange equation is

$$\frac{d}{d\tau} \left(g_{\theta\theta} \dot{\theta} \right) - \frac{1}{2} g_{\mu\nu,\theta} \dot{x}^\mu \dot{x}^\nu = 0 \quad (31)$$

Reflection symmetry gives

$$g_{\mu\nu,\theta}|_{\pi/2} = 0$$

Expanding Eq. (31) to first order in η gives

$$\eta + \omega_\theta^2 \eta = 0 \quad (32)$$

where the proper-time vertical frequency is

$$\omega_\theta^2 = - \frac{1}{2g_{\theta\theta}} \left(g_{tt,\theta\theta} \dot{t}^2 + g_{\varphi\varphi,\theta\theta} \dot{\varphi}^2 \right) \Big|_{r_0,\pi/2} \quad (33)$$

In the present metric g_{tt} is independent of θ , so only the azimuthal term contributes. Dividing by $(u^t)^2$ gives the coordinate-time frequency:

$$\Omega_\theta^2 = - \frac{g_{\varphi\varphi,\theta\theta}}{2g_{\theta\theta}} \Omega^2 \Big|_{r_0,\pi/2} \quad (34)$$

For Eqs. (5) and (6),

$$g_{\varphi\varphi,\theta\theta}|_{\pi/2} = 2r^2(1 - q \ln f) \quad (35)$$

whereas

$$g_{\theta\theta}|_{\pi/2} = -r^2[1 + q \ln f - 2q \ln H_{eq}] \quad (36)$$

It follows that

$$\Omega_\theta^2 = \Omega^2 \left[1 + 2q \ln \left(\frac{H_{eq}}{f} \right) \right] + O(q^2) \quad (37)$$

Using $H_{eq} = (1 - m/r)^2$, the ratio becomes

$$\frac{\Omega_\theta^2}{\Omega^2} = 1 + 2q \ln \left[\frac{(1 - m/r)^2}{1 - 2m/r} \right] + O(q^2) \quad (38)$$

Taking the positive root,

$$\frac{\Omega_\theta}{\Omega} = 1 + q \ln \left[\frac{(1 - m/r)^2}{1 - 2m/r} \right] + O(q^2) \quad (39)$$

This compact expression is the main analytical result of the conservative part of the paper.

Combining Eqs. (25) and (37) gives:

$$\Omega_\theta^2 = \frac{m}{r^3} \left\{ 1 + q \left[1 + 2 \ln H_{eq} + \frac{m}{r - 2m} \right] \right\} + O(q^2) \quad (40)$$

At fixed physical mass, the corresponding expression is:

$$\Omega_\theta^2 = \frac{M}{r^3} \left\{ 1 + q \left[2 \ln \left(1 - \frac{M}{r} \right)^2 + \frac{M}{r - 2M} \right] \right\} + O(q^2) \quad (41)$$

Vertical stability condition

The circular orbit is vertically stable when:

$$\Omega_\theta^2(r_0, q) > 0 \quad (42)$$

Within the first-order treatment this condition must be used together with Eq. (9). A formal sign change obtained where the correction is as large as the zeroth-order term would lie outside the controlled perturbative regime and should not be interpreted as a reliable instability boundary.

For $r > 2m$:

$$\frac{(1-m/r)^2}{1-2m/r} = 1 + \frac{m^2}{r(r-2m)} > 1 \quad (43)$$

he logarithm in Eq. (39) is therefore positive. At fixed m and within the valid weak-deformation domain, positive q increases Ω_θ/Ω , while negative q decreases it. This statement concerns the ratio of frequencies and is not affected by an overall rescaling of the mass.

Vertical stability is not the only requirement for a long-lived circular orbit. Radial stability must also be checked. Since the present article focuses on vertical motion, we restrict the numerical examples to radii safely outside the Schwarzschild marginally stable value and away from any perturbatively shifted radial boundary.

We therefore use $r_\phi = 8M$ for the settling examples. The frequency curves begin at $r=6M$ only to display the radial trend; no claim about radial stability at every point of those curves is made here.

Quadrupole-induced frequency splitting

In Schwarzschild spacetime, spherical symmetry gives:

$$\Omega_\theta = \Omega \quad (44)$$

The q -deformation removes this degeneracy. Defining:

$$\Delta_\theta = \frac{\Omega_\theta - \Omega}{\Omega} \quad (45)$$

we find:

$$\Delta_\theta = q \ln \left[\frac{(1-m/r)^2}{1-2m/r} \right] + O(q^2) \quad (46)$$

The related vertical-precession frequency may be defined as:

$$\Omega_{vp} = \Omega - \Omega_\theta \quad (47)$$

so that

$$\frac{\Omega_{vp}}{\Omega} = -q \ln \left[\frac{(1-m/r)^2}{1-2m/r} \right] + O(q^2) \quad (48)$$

There is no frame dragging in this static spacetime. The splitting in Eq. (48) is therefore caused by the quadrupolar departure from spherical symmetry rather than by rotation.

At large radius,

$$\ln \left[\frac{(1-m/r)^2}{1-2m/r} \right] = \frac{m^2}{r^2} + O\left(\frac{m^3}{r^3}\right) \quad (49)$$

and the relative splitting decays as qm^2/r^2 . It is consequently a strong-field correction, although the first-order metric must not be pushed too close to $r=2m$.

Validity of the first-order approximation

The Schwarzschild case $q=0$ remains the physical baseline for the frequency shift. The unexpanded q -metric is introduced here only as an internal validation tool: it measures the error made by dropping $O(q^2)$ and higher terms without requiring an orbit integration.

For the exact q -metric, the equatorial components relevant here are

$$A_{ex} = f^{1+q}, C_{ex} = r^2 f^{-q} \quad (50)$$

Equation (19) then gives

$$\Omega_{ex}^2 = \frac{m(1+q)}{r^3} f^{2q} \frac{r-2m}{r-(2+q)m} \quad (51)$$

The exact vertical-to-orbital ratio is

$$\frac{\Omega_{\theta,ex}^2}{\Omega_{ex}^2} = \left[\frac{(1-m/r)^2}{1-2m/r} \right]^{q(2+q)} \quad (52)$$

Expanding Eqs. (51) and (52) around $q=0$ reproduces Eqs. (25) and (38). The exact expressions will only be used to estimate the truncation error,

$$\epsilon_X(r, q) = \left| \frac{X_{O(q)} - X_{ex}}{X_{ex}} \right|, X \in \left\{ \Omega, \Omega_{\theta}, \frac{\Omega_{\theta}}{\Omega} \right\} \quad (53)$$

This comparison does not change the perturbative character of the model, it simply shows where the first-order formulas remain accurate.

For the numerical error estimates, the physical monopole mass M is held fixed. Accordingly, the metric parameter entering the exact q-metric expressions is set to $m=M/(1+q)$. The exact orbital frequency is evaluated from Eq. (51), while the exact vertical frequency is obtained by combining Eqs. (51) and (52). The first-order values are calculated from the fixed- M expressions derived above. The tabulated percentage error is

$$\epsilon^{(\%)}_X(r, q) = 100 \left| \frac{X_{O(q)} - X_{ex}}{X_{ex}} \right|, X \in \left\{ \Omega_{\theta}, \frac{\Omega_{\theta}}{\Omega} \right\}$$

Table 1 reports this quantity for $q=\pm 0.10$ at $r/M=6, 8$, and 10 . All digits displayed in the table are obtained from the unrounded frequency values; rounding is applied only to the final percentages.

Dissipative settling

Phenomenological damping model

The geodesic equation (32) conserves the vertical oscillation amplitude. To model weak energy loss, we introduce a linear damping coefficient in proper time,

$$\ddot{\eta} + 2\Gamma\dot{\eta} + \omega_{\theta}^2\eta = 0, \Gamma \geq 0 \quad (54)$$

This equation is phenomenological. It represents the local linear response of an unspecified dilute medium or weak collisional process.

The displacement η is dimensionless, whereas Γ has the dimension of inverse proper time. In geometrized units, it therefore has the dimension of inverse length or inverse mass, and ΓM is dimensionless. The coefficient Γ measures the local fractional decay rate of the oscillation amplitude per unit proper time. It is introduced independently of the vacuum q-metric and is not derived from a specific microscopic interaction. The model may represent the leading linear response of a dilute medium, weak collisions, or a sufficiently weak radiation-drag force. It is not intended as a complete model of an accretion disc.

The damping term can also be written using coordinate time when the background circular orbit is fixed. Since γ is constant at the reference orbit, Eq. (54) becomes

$$\frac{d^2\eta}{dt^2} + 2\gamma\frac{d\eta}{dt} + \Omega_{\theta}^2\eta = 0, \gamma = \frac{\Gamma}{u^t} \quad (55)$$

The coordinate-time coefficient γ has the dimension of inverse coordinate time, so that γM is dimensionless. Equation (55) also distinguishes two physically different comparison prescriptions. If Γ is held fixed, all configurations have the same decay envelope in proper time, $A_{\text{env}}(\tau) = A_0 e^{-\Gamma\tau}$, while the corresponding coordinate-time rate $\gamma = \Gamma/u^t(r, q)$ varies with the gravitational redshift and therefore generally depends on r and q . If γ is held fixed instead, all trajectories have the same coordinate-time envelope, $A_{\text{env}}(t) = A_0 e^{-\gamma t}$, but the associated proper-time coefficient

$\Gamma = u'(r, q)\gamma$ is not identical for different values of q . The numerical trajectories presented below use the second prescription because they are compared at the same orbital radius in the observer's coordinate time.

Analytical solution

For weak damping,

$$0 \leq \Gamma < \omega_\theta \quad (56)$$

Mathematically, $\Gamma \geq 0$ and $\gamma \geq 0$. Oscillatory settling requires the underdamped conditions $\Gamma < \omega_\theta$ and, equivalently, $\gamma < \Omega_\theta$. The weak-damping regime considered here is more restrictive, $\Gamma/\omega_\theta \ll 1$, or $\gamma/\Omega_\theta \ll 1$. In the numerical examples, $\gamma M = 0.002$, while $M\Omega_\theta \simeq 0.043\text{--}0.045$, giving $\gamma/\Omega_\theta \simeq 0.044\text{--}0.046$. The adopted value therefore lies safely within the underdamped regime and produces several oscillation cycles before the amplitude reaches the specified settling threshold.

The proper-time solution is

$$\eta(\tau) = A_0 e^{-\Gamma\tau} \cos(\omega_d \tau + \delta_0), \quad \omega_d = \sqrt{\omega_\theta^2 - \Gamma^2} \quad (57)$$

In coordinate time,

$$\eta(t) = A_0 e^{-\gamma t} \cos(\Omega_d t + \delta_0), \quad \Omega_d = \sqrt{\Omega_\theta^2 - \gamma^2} \quad (58)$$

The Schwarzschild conservative limit is recovered by setting $q=0$ and $\Gamma=0$. Setting only $\Gamma=0$ gives persistent q -dependent vertical oscillations.

For completeness, critical damping occurs at $\Gamma = \omega_\theta$, while $\Gamma > \omega_\theta$ gives overdamped motion. Those regimes are not the main focus here because the assumed physical situation is weak dissipation over several orbital cycles.

Settling diagnostics

The phrase “settling time” needs a specified threshold. We apply that threshold to the oscillation envelope, not to an instantaneous equatorial crossing. Thus, the particle is considered settled when

$$A_{env}(\tau) = A_0 e^{-\Gamma\tau} \leq \eta_f, \quad 0 < \eta_f < A_0 \quad (59)$$

The proper settling time is then

$$\tau_{set} = \frac{1}{\Gamma} \ln \left(\frac{A_0}{\eta_f} \right) \quad (60)$$

For a constant Γ , this time has no explicit q dependence. The result follows from the chosen damping law: Γ fixes the decay envelope, whereas the geometry fixes the oscillation frequency inside that envelope.

The quadrupole affects the number of vertical cycles completed before the threshold is reached. In the weak-damping limit,

$$N_{set} \simeq \frac{\omega_d \tau_{set}}{2\pi} = \frac{\omega_d}{2\pi\Gamma} \ln \left(\frac{A_0}{\eta_f} \right) \quad (61)$$

It is also useful to measure settling in orbital periods,

$$N_{orb} \simeq \frac{\Omega t_{set}}{2\pi} = \frac{\Omega}{2\pi\gamma} \ln \left(\frac{A_0}{\eta_f} \right) \quad (62)$$

The ratio

$$\frac{N_{set}}{N_{orb}} \simeq \frac{\Omega_\theta}{\Omega} \quad (63)$$

directly carries the quadrupole correction in Eq. (39).

Another simple diagnostic is the linearized vertical energy,

$$E_\theta = \frac{1}{2} \dot{\eta}^2 + \frac{1}{2} \omega_\theta^2 \eta^2 \quad (64)$$

Equation (54) gives

$$\frac{dE_\theta}{d\tau} = -2\Gamma\eta^2 \leq 0 \quad (65)$$

Thus $\omega_\theta(r, q)$ sets the restoring part of the motion, while Γ accounts for the irreversible loss of vertical energy.

Research Results

Numerical illustrations

Only a small set of plots is needed because the central results are analytical. We use the physical mass M as the unit of length and time and take the symmetric weak-deformation sample

$$q \in \{-0.10, -0.05, 0, 0.05, 0.10\} \quad (66)$$

over the interval $6 \leq r/M \leq 20$. In this domain both logarithmic corrections in Eq. (9) remain small. The lower boundary is not used as a claim about the perturbed ISCO, it only marks the beginning of the illustrative interval. The damped trajectories below are evaluated at $r_0 = 8M$, safely away from that boundary.

All numerical illustrations are obtained by direct evaluation of the analytical expressions; no numerical integration of the geodesic or damped-oscillator equations is required. The physical monopole mass M is used as the unit of length and time. In expressions that are already multiplied by q , m may consistently be replaced by M , because the difference $m - M = O(q)$ would contribute only at $O(q^2)$. For comparisons with the exact q-metric, however, we use the exact relation $m = M/(1+q)$. Figures 1 and 2 are evaluated for $q = \{-0.10, -0.05, 0, 0.05, 0.10\}$, $6 \leq r/M \leq 20$.

Figures 3 and 4 use the analytical solution (58) at $r_0 = 8M$, with $A_0 = 0.05$, $\delta_0 = 0$, and $\gamma M = 0.002$, over $0 \leq t/M \leq 1250$. Figure 4 is constructed from the same trajectories using the analytically differentiated velocity dn/dt , normalized by $\Omega_\theta(0) = \Omega_\theta(r_0, q=0)$. The calculations and plotting were performed in Wolfram Mathematica using machine-precision arithmetic.

The plotting routine used adaptive sampling of the closed analytical functions, with additional recursive refinement in regions of appreciable curvature.

Figure 1 shows

$$\frac{\Omega_\theta}{\Omega} = 1 + q \ln \left[\frac{(1-m/r)^2}{1-2m/r} \right] \quad (67)$$

as a function of r/M . It isolates the loss of spherical degeneracy without being dominated by the overall Keplerian decrease of both frequencies.

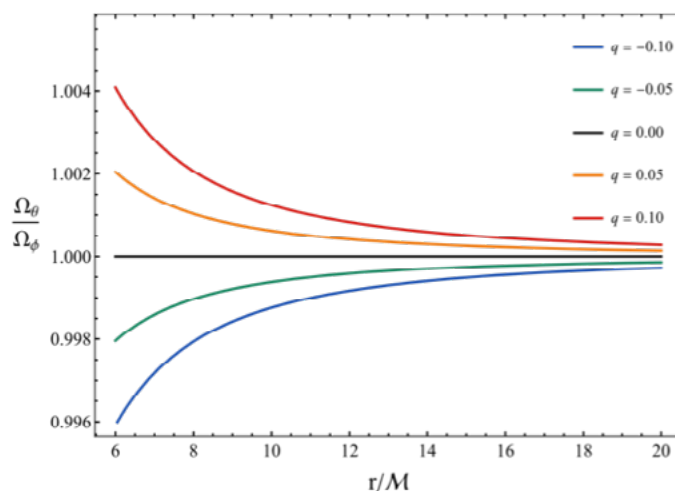


Figure 1. Ratio as a function of for the weak-deformation sample in Eq. (66). The Schwarzschild curve is the spherical baseline; positive and negative shift the ratio to opposite sides of unity.

The second panel shows the normalized vertical-precession frequency Ω_{vp}/Ω . We then compare $\eta(t)$ for several values of q at fixed $r_0=8M$, $A_0=0.05$, $\delta_0=0$, and $\gamma M=0.002$. Here γ , rather than Γ , is deliberately held fixed. All trajectories therefore share the same coordinate-time envelope, while their accumulated phases differ. The final panel shows the corresponding phase portrait.

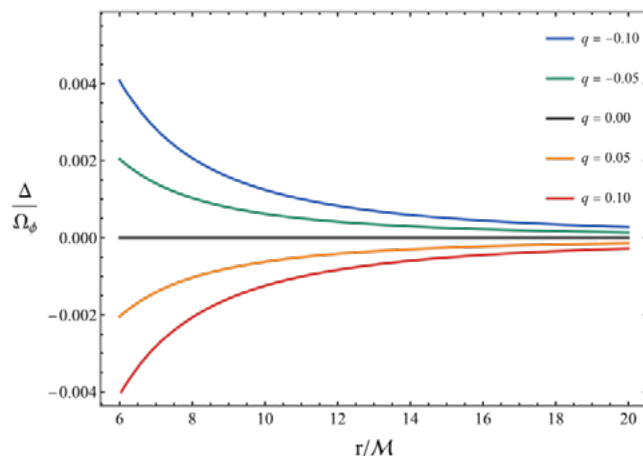


Figure 2. Normalized vertical-precession frequency $\Omega_{vp}/\Omega=(\Omega-\Omega_\theta)/\Omega$. The splitting vanishes for $q=0$ and approaches zero at large radius.

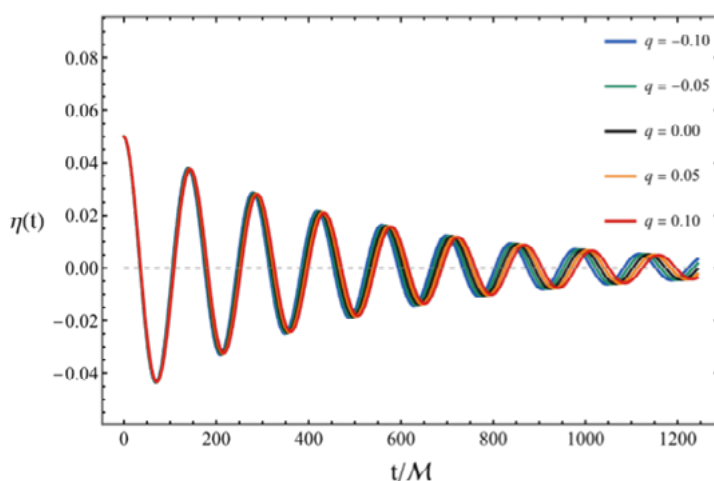


Figure 3. Damped vertical displacement at $r_0=8M$ for $\gamma M=0.002$, $A_0=0.05$, and $\delta_0=0$. Holding γ fixed gives the same exponential envelope for every q , the deformation changes the phase accumulated inside that envelope.

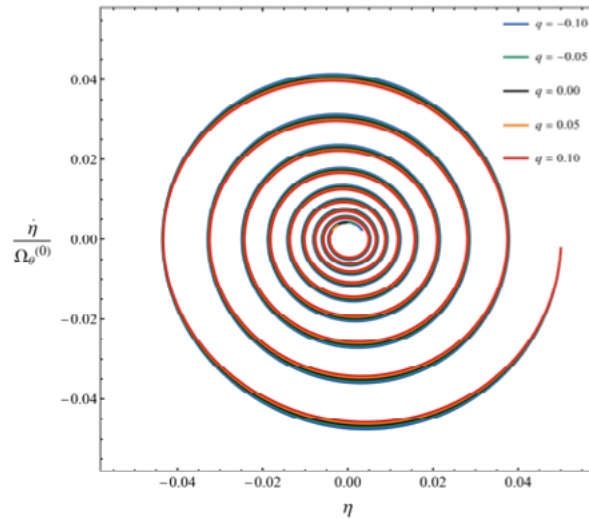


Figure 4. Phase portrait for the trajectories in Fig. 3. The vertical velocity is normalized by $\Omega_\theta^{(0)} \equiv \Omega_\theta(r_0, q=0)$. The inward spirals show the loss of vertical energy in the phenomenological damping model.

Table 1 compares the first-order and exact formulas at the ends of the chosen deformation interval. The largest error in Ω_θ is about 0.33% and occurs for $q=-0.1$ at $r=6M$. The ratio Ω_θ/Ω , which is the main frequency diagnostic, is more accurate and differs from the exact value by less than 0.10% in the same sample.

Table 1. Relative error of the first-order expressions with respect to the exact q-metric at fixed physical mass. Errors are given in percent.

q	r/M	ϵ_{Ω_θ}	$\epsilon_{\Omega_\theta/\Omega}$
-0.10	6	0.332	0.095
-0.10	8	0.247	0.044
-0.10	10	0.193	0.025
0.10	6	0.284	0.067
0.10	8	0.208	0.032
0.10	10	0.162	0.019

For the fixed- γ illustration, the envelope reaches $\eta_f=0.005$ at $t_{set}=1151.29M$. This time is common to all curves because their coordinate-time decay law is the same. The number of vertical cycles is not: it decreases from 8.237 at $q=-0.10$ to 7.939 at $q=0.10$. This small difference is the accumulated effect of the quadrupole-induced frequency shift.

Table 2. Frequency and settling diagnostics at $r_0=8M$ for $\gamma M=0.002$, $A_0=0.05$, and $\eta_f=0.005$.

q	$M\Omega$	$M\Omega_\theta$	N_{set}
-0.10	0.045088	0.044999	8.237
-0.05	0.044643	0.044598	8.164
0	0.044194	0.044194	8.090
0.05	0.043740	0.043786	8.015
0.10	0.043282	0.043375	7.939

Discussion and conclusions

We have examined vertical motion near equatorial circular orbits in the q -metric, working to first order in the deformation parameter. The spherical case $q=0$ serves as the physical reference throughout, while the exact q -metric is used to assess the accuracy of the linear approximation. The main conservative result is Eq. (39), which describes the $O(q)$ splitting between the vertical epicyclic and orbital frequencies at fixed physical monopole mass.

With the convention adopted here, the sign of the shift is determined by the logarithmic term in Eq. (39). This term is positive for $r > 2m$, so that positive q increases Ω_θ/Ω , while negative q decreases it. In the far-field limit, the splitting falls off as qm^2/r^2 and becomes small. The first-order expansion is not reliable arbitrarily close to $r = 2m$, however, because the logarithmic corrections cease to be perturbative there.

A stable vertical perturbation remains oscillatory in the conservative system; the quadrupolar geometry alone does not cause the orbit to settle onto the equatorial plane. Irreversible decay appears only after the phenomenological damping term is added. For a constant proper-time damping coefficient, the amplitude decays at the corresponding proper-time rate. In the numerical examples, we instead hold the coordinate-time coefficient γ fixed. The resulting trajectories therefore have the same coordinate-time decay envelope, although their oscillation frequencies differ.

The deformation consequently changes the phase accumulation and the number of vertical cycles completed before a prescribed near-equatorial threshold is reached. It does not alter the imposed exponential envelope at fixed γ . This distinction is important: the q -metric supplies the restoring frequency, whereas dissipation enters as a separate physical assumption.

The analysis is deliberately local and has several limitations. The damping law is phenomenological, radial motion is frozen at linear order, and the exterior q -metric is not matched to a particular material source. The model therefore does not describe the formation of an accretion disc or a self-gravitating ring. In a more realistic setting, the drag coefficient would depend on the properties of the surrounding medium and on the particle's relative velocity. Allowing simultaneous radial drift would also make the vertical frequency evolve along the trajectory, changing both the decay history and the accumulated phase.

A medium-specific drag law would generally replace the constant coefficient by a function such as $\Gamma = \Gamma(r, q, \rho, v_{rel}, \dots)$, where ρ and v_{rel} characterize the local medium and the particle's motion relative to it. The decay envelope would then be determined by $A_{env}(\tau) = A_0 e^{-\int \Gamma(\tau) d\tau}$, and the settling condition would become $\int_0^\tau \Gamma(\tau) d\tau = \ln(A_0/\eta_f)$. Consequently, the settling time could acquire an additional dependence on q through the gravitational redshift, the orbital velocity, the properties of the surrounding medium, or a simultaneous radial drift. Such model-dependent effects are intentionally excluded from the present minimal treatment, in which the geometrical frequency shift is isolated from the independently prescribed damping rate.

General curvature-based relations for epicyclic frequencies are already known [20, 21], and exact frequency expressions for the γ -metric have been reported previously. The present calculation adds a fixed-physical-mass form of the weak-deformation frequency shift, identifies Ω_θ/Ω as a scale-independent measure of the splitting, and tests the accuracy of the $O(q)$ approximation directly against the exact q -metric. It also shows how this conservative shift enters the settling diagnostics of Eqs. (61)–(63). At the points sampled in Table 1, the differences between the linear and exact results remain within the reported bounds.

These first-order expressions provide a useful reference for weakly deformed q -metric configurations. A natural extension would be to replace the constant damping coefficient by a drag law derived from a specified matter or radiation environment and to evolve the radial and vertical motions simultaneously.

Acknowledgments

This research was funded by the Ministry of Science and Higher Education of the Republic of Kazakhstan (Zhas Galym 2025–2027, Grant No. AP25795219).

Author contributions

A. Muratkhan — performed the analytical calculations, analyzed the results, and prepared the manuscript.

A.S. Taukenova — developed the research methodology, verified the calculations, and scientifically edited the text.

A. Dalelkhan — performed the numerical calculations and prepared the graphs and tables.

A.T. Orazymbet — developed the research concept, coordinated the work, and edited the final version of the manuscript.

Conflict of Interest

The authors declare that they have no conflict of interest related to the publication of this article.

References

1. D. M. Zipoy, *Topology of some spheroidal metrics*, J. Math. Phys. 7, 1137–1143 (1966). <https://doi.org/10.1063/1.1705005>.
2. B. H. Voorhees, *Static axially symmetric gravitational fields*, Phys. Rev. D 2, 2119–2122 (1970). <https://doi.org/10.1103/PhysRevD.2.2119>.
3. H. Quevedo, S. Toktarbay, and Y. Aimuratov, *Quadrupolar gravitational fields described by the q -metric*, Int. J. Math. Phys. 3(2), 133–139 (2012). <https://ijmph.kaznu.kz/index.php/kaznu/article/view/56>.
4. F. Frutos-Alfaro, H. Quevedo, and P. A. Sánchez, *Comparison of vacuum static quadrupolar metrics*, R. Soc. Open Sci. 5, 170826 (2018). <https://doi.org/10.1098/rsos.170826>.
5. K. S. Virbhadra, *Directional naked singularity in general relativity*, arXiv/9606004 (1996). <https://arxiv.org/abs/gr-qc/9606004>.
6. Z. Stuchlík and J. Schee, *Optical effects in the external field of the gamma metric*, Class. Quantum Grav. 30, 075012 (2013). <https://doi.org/10.1088/0264-9381/30/7/075012>.
7. K. Boshkayev, E. Gasperín, A. C. Gutiérrez-Piñeres, H. Quevedo, and S. Toktarbay, *Motion of test particles in the field of a naked singularity*, Phys. Rev. D 93, 024024 (2016). <https://doi.org/10.1103/PhysRevD.93.024024>.
8. S. Toktarbay, H. Quevedo, M. Abishev, and A. Muratkhan, *Gravitational field of slightly deformed naked singularities*, Eur. Phys. J. C 82, 382 (2022). <https://doi.org/10.1140/epjc/s10052-022-10230-2>.
9. Idrissov, K. Boshkayev, K. F. Dialektopoulos, A. Urazalina, and D. Utepova, *Geodesic deviation in the q -metric*, Eur. Phys. J. C 85, 319 (2025). <https://doi.org/10.1140/epjc/s10052-025-13969-6>.
10. N. Aliev and D. V. Gal'tsov, *Radiation from relativistic particles in nongeodesic motion in a strong gravitational field*, Gen. Relativ. Gravit. 13, 899–912 (1981). <https://doi.org/10.1007/BF00756068>.
11. T. Okazaki, S. Kato, and J. Fukue, *Global trapped oscillations of relativistic accretion disks*, Publ. Astron. Soc. Jpn. 39, 457–473 (1987). <https://doi.org/10.1093/pasj/39.3.457>.
12. M. A. Nowak and R. V. Wagoner, *Diskoseismology: Probing accretion disks. I. Trapped adiabatic oscillations*, Astrophys. J. 353, 247–266 (1990). <https://doi.org/10.1086/168612>.
13. M. A. Abramowicz and W. Kluźniak, *A precise determination of black hole spin in GRO J1655–40*, Astron. Astrophys. 374, L19–L20 (2001). <https://doi.org/10.1051/0004-6361:20010791>.
14. W. Kluźniak and M. A. Abramowicz, *Strong-field gravity and orbital resonance in black holes and neutron stars—kHz quasi-periodic oscillations (QPO)*, Acta Phys. Pol. B 32, 3605–3612 (2001).
15. G. Török and Z. Stuchlík, *Radial and vertical epicyclic frequencies of Keplerian motion in the field of Kerr naked singularities*, Astron. Astrophys. 437, 775–788 (2005). <https://doi.org/10.1051/0004-6361:20052825>.
16. P. J. Montero, O. Zanotti, J. A. Font, and L. Rezzolla, *Oscillations of vertically integrated relativistic tori—I. Axisymmetric modes in a Schwarzschild spacetime*, Mon. Not. R. Astron. Soc. 354, 1040–1052 (2004). <https://doi.org/10.1111/j.1365-2966.2004.08265.x>.
17. E. Rubio-Herrera and W. H. Lee, *Oscillations of thick accretion discs around black holes—II. Effect of the angular momentum distribution*, Mon. Not. R. Astron. Soc. 362, 789–798 (2005). <https://doi.org/10.1111/j.1365-2966.2005.09357.x>.
18. S. Faraji, A. Trova, and H. Quevedo, *Relativistic equilibrium fluid configurations around rotating deformed compact objects*, Eur. Phys. J. C 82, 1149 (2022). <https://doi.org/10.1140/epjc/s10052-022-11075-5>.

19. S. Faraji, *The impact of compact object deformation on thin accretion disk properties*, Eur. Phys. J. C 85, 148 (2025). <https://doi.org/10.1140/epjc/s10052-025-13794-x>.
20. R. S. S. Vieira, W. Kluźniak, and M. A. Abramowicz, *Curvature dependence of relativistic epicyclic frequencies in static, axially symmetric spacetimes*, Phys. Rev. D 95, 044008 (2017). <https://doi.org/10.1103/PhysRevD.95.044008>.
21. J. F. M. Delgado, C. A. R. Herdeiro, and E. Radu, *Epicyclic frequencies for a generic ultracompact object*, Phys. Rev. D 106, 064054 (2022). <https://doi.org/10.1103/PhysRevD.106.064054>.
22. F. D. Ryan, *Gravitational waves from the inspiral of a compact object into a massive, axisymmetric body with arbitrary multipole moments*, Phys. Rev. D 52, 5707–5718 (1995). <https://doi.org/10.1103/PhysRevD.52.5707>.
23. G. Pappas and T. P. Sotiriou, *Geodesic properties in terms of multipole moments in scalar–tensor theories of gravity*, Mon. Not. R. Astron. Soc. 453, 2862–2876 (2015). <https://doi.org/10.1093/mnras/stv1819>.
24. Toshmatov, D. Malafarina, and N. Dadhich, *Harmonic oscillations of neutral particles in the γ -metric*, Phys. Rev. D 100, 044001 (2019). <https://doi.org/10.1103/PhysRevD.100.044001>.
25. Bini, R. T. Jantzen, and L. Stella, *The general relativistic Poynting–Robertson effect*, Class. Quantum Grav. 26, 055009 (2009). <https://doi.org/10.1088/0264-9381/26/5/055009>.
26. V. De Falco, P. Bakala, E. Battista, D. Lančová, M. Falanga, and L. Stella, *The three-dimensional general relativistic Poynting–Robertson effect I: Radial radiation field*, Phys. Rev. D 99, 023014 (2019). <https://doi.org/10.1103/PhysRevD.99.023014>.

А. Мұратхан¹, А.С. Тәукенова¹, А. Дәлелхан², А.Т. Оразымбет^{*1}

¹Теориялық және ядролық физика кафедрасы, әл-Фараби атындағы

Қазақ ұлттық университеті, Алматы, Қазақстан

²Физика кафедрасы, Қазақ ұлттық қыздар педагогикалық университеті,

Физика кафедрасы, Алматы, Қазақстан

(E-mail: muratkhan.aray@kaznu.kz, liya_tauken@mail.ru, dalelkhankyzy.a@qyzpu.edu.kz,

*ayazhan.orazymbet@kaznu.edu.kz)

Әлсіз деформацияланған q–метрика кеңістік-уақытындағы вертикаль эпициклдік орнықтылық және диссипативтік орнығу

Аңдатпа. Біз әлсіз деформацияланған q–метрика кеңістік-уақытының экваторлық жазықтығына жақын аймақтағы массалы сынақ бөлшектерінің қозғалысын зерттейміз. Өлшемсіз квадруполь параметрі бойынша бірінші реттік жуықтауда жұмыс істей отырып және физикалық монопольдік массаны тұрақты сақтай отырып, экваторлық шеңберлік орбиталар үшін орбиталық бұрыштық жылдамдық пен вертикаль эпициклдік жиілікті қорытып шығарамыз. кезінде Шварцшильд шегі қалпына келеді, ал нөлден өзгеше квадруполь вертикаль және азимуттық жиіліктер арасындағы дегенерацияны деңгейінің өзінде жояды.

Одан кейін ауытқыған орбитаның экваторлық жазықтыққа біртіндеп жақындауын сипаттау үшін сызықтандырылған вертикаль теңдеуге шағын феноменологиялық демпфирлеу мүшесі енгізіледі. Консервативтік жағдайда бөлшек осы жазықтықтың маңында амплитудасы кемімей тербелуін жалғастырады. Демпфирлеу болған кезде амплитуда экспоненциалды түрде кемиді. Квадрупольдік деформация вертикаль тербеліс жиілігін және соның салдарынан берілген орнығу шегіне жеткенге дейін орындалатын циклдер санын өзгертеді. Алайда ол сөнудің уақыт масштабын анықтамайды, өйткені бұл масштаб қабылданған демпфирлеу тәсілінен туындайды. Біздің сандық мысалдарымызда координаталық уақыт бойынша демпфирлеу коэффициенті тұрақты сақталғандықтан, траекториялардың экспоненциалдық ораушысы бірдей, бірақ олардың тербеліс жиіліктері әртүрлі болады.

Дәл q–метрикамен салыстыру бірінші реттік жуықтауды тікелей тексеруге де мүмкіндік береді. Қарастырылған және аймағында вертикаль жиіліктің салыстырмалы қателігі 0.34%-дан аспайды, ал сәйкес жиіліктер қатынасы 0.10%-дан төмен қателікпен беріледі. Бұл нәтижелер диссипативтік орнығу тәуелсіз

түрде енгізілген модель аясында квадрупольдік геометрияның вертикаль орбиталық динамиканы қалай өзгертетінін айқын көрсетеді.

Түйін сөздер: q -метрика; квадрупольдік деформация; шеңберлік геодезиялық орбиталар; вертикаль эпидиклдік жиілік; диссипативтік орнығу.

А. Муратхан¹, А.С. Таукенова¹, А. Дәлелхан², А.Т. Оразымбет^{*1}

¹*Кафедра теоретической и ядерной физики, Казахский национальный университет имени аль-Фараби, Алматы, Казахстан*

²*Кафедра физики, Казахский национальный женский педагогический университет, Алматы, Казахстан*

(E-mail: muratkhan.aray@kaznu.kz, liya_tauken@mail.ru, dalelkhankyzy.a@qyzpu.edu.kz,
*ayazhan.oraizymbet@kaznu.edu.kz)

Вертикальная эпидиклическая устойчивость и диссипативное оседание в пространстве-времени слабо деформированной q -метрики

Аннотация. В данной работе исследуется движение массивных пробных частиц вблизи экваториальной плоскости пространства-времени слабо деформированной q -метрики. В приближении первого порядка по безразмерному квадрупольному параметру, при сохранении фиксированной физической монополярной массы, выводятся орбитальная угловая скорость и вертикальная эпидиклическая частота для экваториальных круговых орбит. При восстановливается предел Шварцшильда, тогда как ненулевой квадруполь устраняет вырождение между вертикальной и азимутальной частотами уже в порядке.

Затем в линеаризованное уравнение вертикального движения вводится малый феноменологический член затухания, предназначенный для описания постепенного приближения возмущённой орбиты к экваториальной плоскости. В консервативном случае частица продолжает совершать колебания относительно этой плоскости без уменьшения амплитуды. При наличии затухания амплитуда уменьшается экспоненциально. Квадрупольная деформация изменяет частоту вертикальных колебаний и, следовательно, количество циклов, совершаемых до достижения заданного порога оседания. Однако она не определяет временной масштаб затухания, который задаётся принятой моделью демпфирования. Поскольку в численных примерах коэффициент демпфирования по координатному времени сохраняется постоянным, траектории имеют одинаковую экспоненциальную огибающую, но колеблются с различными частотами.

Сравнение с точной q -метрикой также обеспечивает непосредственную проверку приближения первого порядка. В исследованной области и относительная погрешность вертикальной частоты остаётся ниже 0.34%, тогда как соответствующее отношение частот воспроизводится с точностью лучше 0.10%. Полученные результаты наглядно показывают, каким образом квадрупольная геометрия изменяет вертикальную орбитальную динамику в рамках модели, в которой диссипативное оседание вводится независимо.

Ключевые слова: q -метрика; квадрупольная деформация; круговые геодезические орбиты; вертикальная эпидиклическая частота; диссипативное оседание.

Information about the authors:

Muratkhan Aray – PhD, Senior Lecturer, Department of Theoretical and Nuclear Physics, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Taukenova Aliya – PhD, Associate Professor, Department of Theoretical and Nuclear Physics, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Dalelkhankyzy Anar – Department of Physics, Kazakh National Women’s Teacher Training University, Almaty, Kazakhstan.

Orazymbet Ayazhan – Master of Science, Lecturer, Researcher, National Nanotechnology Laboratory of Open Type, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Мұратхан Арай – PhD, аға оқытушы, Теориялық және ядролық физика кафедрасы, әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан.

Тәукенова Әлия – PhD, қауымдастырылған профессор, Теориялық және ядролық физика кафедрасы, әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан.

Далелханқызы Анар – Физика кафедрасы, Қазақ ұлттық қыздар педагогикалық университеті, Алматы, Қазақстан.

Оразымбет Аяжан – жаратылыстану ғылымдарының магистрі, оқытушы, Ашық түрдегі ұлттық нанотехнологиялық зертхананың ғылыми қызметкері, әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан.

Муратхан Арай – PhD, старший преподаватель кафедры теоретической и ядерной физики, Казахский национальный университет имени аль-Фараби, Алматы, Казахстан.

Таукенова Алия – PhD, ассоциированный профессор кафедры теоретической и ядерной физики, Казахский национальный университет имени аль-Фараби, Алматы, Казахстан.

Далелханқызы Анар – кафедра физики, Казахский национальный женский педагогический университет, Алматы, Казахстан.

Оразымбет Аяжан – магистр естественных наук, преподаватель, научный сотрудник национальной нанотехнологической лаборатории открытого типа, Казахский национальный университет имени аль-Фараби, Алматы, Казахстан.



Copyright: © 2026 by the authors. Submitted for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY NC) license (<https://creativecommons.org/licenses/by-nc/4.0/>).