Л.Н. Гумилев атындағы Еуразия ұлттық университетінің ХАБАРШЫСЫ.
ISSN: 2616-6836. eISSN: 2663-1296

АСТРОНОМИЯ СЕРИЯСЫ/ ASTRONOMY SERIES / СЕРИЯ АСТРОНОМИЯ

IRSTI: 41.29.15, 41.29.25
Scientific article

https://doi.org/10.32523/2616-6836-2025-153-4-7-19

Fixed point analysis with steep exponential potential

Muskan Khan, Mohd Shahalam

Department of Physics, Integral University, Lucknow, 226026, India

*(E-mail: *mohdshahamu@gmail.com, mk04567432@gmail.com)*

Abstract. In this work, we explore the dynamical behavior of a cosmological model governed by a steep exponential potential. Our study concentrates on the autonomous form of the evolution equations, enabling us to investigate the asymptotic characteristics of the model. We construct a set of first-order differential equations which is associated with the scalar field and the expansion of the universe. A key part of the analysis involves identifying the critical points of the cosmological model. We determine the stability properties of these points through eigenvalues, classifying them as stable, unstable or saddle points. This classification provides valuable insight into how the universe may evolve qualitatively under the influence of a steep exponential potential. Our results show that for specific parameter ranges, the model admits a stable critical point, which functions as a late-time attractor. This indicates that exponential potential can drive the universe toward a stable phase of evolution. Additionally, we use phase space diagrams to illustrate the trajectories and stability structure of the system, offering a clear visualization of the scalar field domination at the current epoch.

Keywords: Dynamical analysis, Fixed point, Quintessence, Cosmic acceleration, Phase space

Introduction

Astronomy is one of humanity's oldest fields of knowledge, devoted to the study of celestial objects such as stars, galaxies, planets, and moons. A closely related discipline, cosmology, focuses on understanding the Universe as a whole. It applies the scientific method to investigate the origin, evolution, and ultimate fate of the cosmos. Cosmological theories are developed to make specific predictions about observable phenomena, which can be tested through astronomical observations. Depending on the outcomes, these theories may be refined, revised, or replaced. The prevailing explanation for the origin and evolution of the Universe is the Big Bang theory, widely known as the standard cosmological model. This model is grounded in

Received 22.11.2025. Revised 29.11.2025. Accepted 01.12.2025. Available online 25.12.2025

*the corresponding author

Einstein's general theory of relativity, published in 1915. Observational breakthroughs in the 1920s – such as Edwin Hubble's discovery that the Milky Way is just one of many galaxies and Vesto Slipher's early measurements indicating the Universe's expansion – paved the way for modern cosmological models. Soon after, Georges Lemaître and others formulated the Big Bang hypothesis based on general relativity [1,2,3,4,5]. The Big Bang model remains the dominant framework in cosmology and has passed several major observational tests. These include the Hubble diagram describing the Universe's expansion, the predicted abundances of light elements such as deuterium and helium from Big Bang Nucleosynthesis (BBN), and the observation of the cosmic microwave background (CMB), which is the relic radiation from the early Universe. Since the late 20th century, the fields of cosmology and space science have undergone profound developments. One of the most significant discoveries was that the expansion of the Universe is accelerating, implying the existence of a mysterious component known as dark energy. Another major theoretical advancement is cosmic inflation, which posits a brief period of extremely rapid expansion immediately after the Big Bang, helping to explain the large-scale structure of the cosmos.

Astrophysicists generally agree that all matter – ranging from galaxies and stars to planets and life - originated roughly 13 billion years ago from an extremely hot, dense state. Instead of occurring within pre-existing space, the Big Bang created both matter and space itself. In its earliest moments, the Universe was filled with a hot plasma of fundamental particles such as quarks and photons. As expansion progressed, the energy density decreased and the temperature dropped, eventually allowing atoms and more complex structures to form. This cooling process can be loosely compared to the temperature drop experienced when a bottle of carbonated liquid is opened and gas escapes [6,7,8,9,10]. A transformative moment in modern cosmology occurred in 1998, when two independent research groups studying distant Type Ia supernovae reported that the Universe's expansion is accelerating. Before this discovery, cosmologists expected the expansion to slow down due to gravitational attraction, and the parameter describing the second derivative of the expansion – q – was therefore termed the "deceleration parameter." The discovery of acceleration fundamentally changed our understanding of cosmic dynamics. The cosmological constant, Λ, offers a simple explanation for this accelerated expansion, though its history in physics has been complex, having been introduced, discarded, and revived multiple times. Subsequent observations – including detailed supernova studies, measurements of the CMB, large-scale structure surveys, and galaxy cluster data – have confirmed the presence of accelerated expansion with increasing confidence. According to the standard cosmological model, the Universe contains substantial amounts of dark matter (DM) and dark energy (DE), although their true nature remains unknown. Measurements from the Planck satellite indicate that the Universe is composed of approximately 68.3% dark energy, 26.8% dark matter, and 4.9% baryonic (ordinary) matter.

Dynamics of Scalar Field

It is well understood that Einstein's field equations govern the dynamics of the Universe. The left-hand side of these equations encodes the curvature of spacetime, while the righthand side contains the energy–momentum tensor, which represents the matter – energy content of the Universe. In their original form, Einstein's equations do not naturally yield late-time cosmic acceleration. To account for the observed acceleration, one must either introduce an exotic fluid with sufficiently negative pressure or modify the gravitational theory itself. The cosmological constant (Λ), corresponding to the ΛCDM model with an equation-of-state parameter w=−1, is the simplest and most widely studied dark energy candidate. Although the ΛCDM model is consistent with current observations, it faces significant theoretical challenges, notably the cosmological constant and coincidence problems. While cosmological data support the existence of Λ, reconciling its observed value with theoretical predictions remains a major difficulty. This motivates the exploration of alternative explanations for dark energy. Scalar fields naturally arise in high-energy physics, including particle physics and string theory, and provide promising candidates for dark energy. In standard cosmology, the matter content is described by a perfect fluid with a barotropic equation of state, where the pressure p(t) and energy density ρ(t) are functions of cosmic time. The conservation of the energy–momentum tensor, guaranteed by the Bianchi identities, ensures the internal consistency of general relativity and its covariant framework. Observations indicate that the Universe is homogeneous and isotropic on large scales (greater than approximately 100 Mpc). These assumptions underpin the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, which embodies the cosmological principle. Consequently, the scale factor a(t), depending only on cosmic time, fully characterizes the expansion history of the Universe. By integrating the relevant second-order differential equation governing a(t), one obtains a corresponding first-order differential equation that this function must satisfy.

$$3H^2 = \rho_{eff}(a) = \rho_{0m}a^{-3}(t) + \Lambda - \frac{3k}{a^2} \tag{1}$$

$$V(a) = \frac{-\rho_{eff}(a)a^2}{6} \tag{2}$$

Here, *H is the Hubble parameter*, Λ is the cosmological constant, k= 0, ±1 is the curvature constant and the covariant conservation condition of the energy-momentum tensor simplifies to

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \tag{3}$$

This Equation is known as Friedmann equation. It is clear from the motion of a unit mass particle in a potential V = V (a).

$$V(a) = \frac{-\rho_{eff}(a)a^2}{6} \tag{4}$$

On the zero energy position, this leads to:

$$\frac{\dot{a}^2}{2} + v(a) = 0 \tag{5}$$

Where ρeff acts as an total energy viscosity parameterized by the expansion factor a(t). The cosmological constant and curvature term in Eq. (1) can be explained as a technical type of fluid: $p\Lambda = -\rho\Lambda = -\Lambda$ and $pk = -1/3$, $\rho k = -3k/a^2$ Thus, the standard model of cosmology can be represented as a dynamical system to Newtonian mechanics: $\ddot{a} = \partial v/\partial a$ where expansion factor plays the significant role of a positional variable for a hypothetical particle of unit mass bluffing the expanding nature of the macrocosm.

It is worth noting that if the cosmological constant is interpreted as the energy density of the quantum vacuum, then this energy remains constant throughout the evolution of the Universe. The standard cosmological model is characterized by several key features, including the following:

(i) The theory contains inherent limitations and energetic boundaries that highlight the constraints imposed by classical singularities. In describing the structure and evolution of the Universe, we employ the classical framework of general relativity, while regimes beyond the Planck epoch remain outside the scope of our analysis.

(ii) The investigation of universal properties relies on a set of fundamental parameters. These include various density parameters and quantities related to the cosmic microwave background (CMB) spectrum, both of which play essential roles in modern cosmology.

Phase-space analysis using dynamical systems has been extensively explored in the literature. Here, we briefly outline the dynamical systems approach, which is essential for understanding the asymptotic behavior of cosmological models and is broadly classified within the category of autonomous systems [11, 12]. In such systems, the choice of dimensionless variables is motivated by several considerations:

(a) They allow the dynamical system to be cast in a bounded domain.

(b) They possess clear physical interpretations.

(c) They often introduce a structural harmony in the equations, enabling a reduction in the number of variables and yielding a simplified system for analysis.

For clarity, we restrict our discussion to a system of two first-order differential equations, although the method can be generalized to systems of arbitrary size. We consider the following set of coupled differential equations, which describe the evolution of the variables x(t) and y(t) as:

$$\dot{x} = f(x, y, t) \tag{6}$$

$$\dot{y} = g(x, y, t) \tag{7}$$

The given equations represent a system where f and g are functions of x, y, and t. If f and g do not explicitly depend on time, However these equations are appertained to as an independent system. The analysis of the dynamic geste of the independent system can be conducted as described below. To identify the fixed or critical points, we can set the left-hand side of the independent system equal to zero. In simpler terms, a point (x_c, y_c) is considered a critical point if it meets the following condition.

$$f(x, y)_{(x_c, y_c)} = 0 \tag{8}$$

$$G(x,y)_{(x_c,y_c)} = 0 \tag{9}$$

The point (x_c, y_c) would behave as an attractor when it meets the following condition,

$$\big(x(t),y(t)\big) \longrightarrow (x_c,y_c) fort \longrightarrow \infty \tag{10}$$

Next, let us move to the stability around the stationary point. In this case, we consider small perturbations δx and δy near the critical point as

$$x = x_c + \delta x \tag{11}$$

$$y = y_c + \delta y \tag{12}$$

On substituting equations (11) and (12) into equations (6) and (7), we get first order differential equations,

$$\frac{d}{dN}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = M\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \tag{13}$$

Here N = ln(a) and the matrix M rely upon critical point (xc, yc), and is given by

$$\mathrm{M} = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}(x{=}x_c\ ,y{=}y_c) \tag{14}$$

It has two eigenvalues η1, η2, and the general solution for δx and δy is given as follows

$$\delta x = k_1 e^{\{\eta_1 N\}} + k_2 e^{\{\eta_2 N\}} \tag{15}$$

$$\delta y = k_3 e^{\{\eta_1 N\}} + k_4 e^{\{\eta_2 N\}} \tag{16}$$

Here k_1, k_2, k_3 and k_4 are denoted as integration constants. Therefore, the stability of the fixed points can be depicted by the signs of eigenvalues. Usually, the following groups are used [11, 12, 13]:

$\eta_1 < 0$ and $\eta_2 < 0$ → Stable point

$\eta_1 > 0$ and $\eta_2 > 0$ → Unstable point

$\eta_1 < 0$ and $\eta_2 > 0$ or (η1 > 0 and η2 < 0) → Saddle point

The real parts of η_1 and η_2 are negative and the determinant of the matrix M is negative → Stable spiral.

The fixed point behave as an attractor stable point in case of (a) and (d) whereas in case of (b) and (c) it is not possible.

A wide range of cosmological observations, both direct and indirect, indicate that the Universe is currently undergoing accelerated expansion. Within the framework of Einstein's gravity, this acceleration is commonly attributed to dark energy. The simplest and most widely studied candidate for dark energy is the cosmological constant, denoted by Λ, forming the basis of the ΛCDM model. Despite its success in explaining many observational results, the ΛCDM paradigm faces notable theoretical challenges, including the fine-tuning and cosmic coincidence problems [14, 15]. These issues raise the question of whether dark energy is truly a constant or instead a dynamical entity. Scalar fields play a central role in modern cosmology and are frequently invoked as dynamical dark energy candidates, often referred to as quintessence. A slowly varying scalar-field potential can give rise to negative pressure, making such fields suitable for driving cosmic acceleration. Scalar field models are important because they can describe different evolutionary phases of the Universe. In particular, they provide a framework capable of characterizing both the early and late stages of accelerated expansion. The Lagrangian density for minimally coupled scalar field ϕ with a potential V (ϕ) is given by

$$L = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{17}$$

For the above Lagrangian density, Friedmann and Klein Gordan equations can be written as

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_m\right] \tag{18}$$

$$\dot{H} = \frac{-\kappa^2}{2}\left[\dot{\phi}^2 + (1 + w_m)\rho_m\right] \tag{19}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \tag{20}$$

Let us use the following dimensionless variables

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \lambda = \frac{-V_{,\phi}}{\kappa V}, \Gamma = \frac{VV_{,\phi\phi}}{V_{,\phi}^2} \tag{21}$$

Where $V_{,\phi} \equiv dV/d\phi$. The above equations can be rewritten in the autonomous form

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1 - w_m)x^2 + (1 + w_m)(1 - y^2)] \tag{22}$$

$$\frac{dy}{dN} = \frac{-\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1 - w_m)x^2 + (1 + w_m)(1 - y^2)] \tag{23}$$

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x \tag{24}$$

together with a constraint equation

$$x^2 + y^2 + \frac{\kappa^2\rho_m}{3H^2} = 1 \tag{25}$$

where N= ln(a). The equation of state w_ϕ and the fraction of the energy density Ω_ϕ for the field ϕ is defined as

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{x^2 - y^2}{x^2 + y^2} \tag{26}$$

ISSN: 2616-6836. eISSN: 2663-1296

$$\Omega_\phi = \frac{\kappa^2 \rho_\phi}{3H^2} = x^2 + y^2 \quad (27)$$

We have also introduced the total effective equation of state:

$$w_{\text{eff}} = \frac{p_\phi + p_m}{\rho_\phi + \rho_m} = w_m + (1 - w_m)x^2 - (1 + w_m)y^2 \quad (28)$$

An accelerating expansion occurs for $w_{eff} \leftarrow 1/3$.

The first potential we consider is the steep exponential potential. Exponential potentials arise in a variety of contexts in physics and play an important role in understanding high-energy phenomena such as string theory and inflation. In string and superstring theories, for example, exponential potentials frequently emerge in models involving compactification from higher dimensions to four dimensions. Despite its simplicity, the single exponential potential represents one of the most fundamental quintessence models and possesses a rich dynamical structure with several essential features.

We study the cosmological dynamics of standard and steeper exponential potentials. Let us consider the following scalar field model characterized by the potential.

V(φ) = V₀ e^{α (φ / M_P)^n} (29)

where V_0 is constant of mass dimension 4. For this potential, we have

Γ = 1+ (n-1)/n α (M_P/φ)^n (30)

In this work, we shall choose n=1. The critical points of the autonomous system are given by the subsequent subsection. The existence of stable point is determined after analyzing the condition of point stays within the phase space, i.e. satisfying the Friedmann constraint $x^2+y^2=1$.

We use autonomous system (22), (23) and (24) to find the stationary points (i.e. $\frac{dx}{dn} = 0$, $\frac{dy}{dn} = 0$ and $\frac{d\lambda}{dN} = 0$). Hence,

Table 1. We present stationary points and their stability for steep exponential potential.

S. No.	Critical Points		Eigen Values		Stability
	x	y	η_1	η_2	
A1.	-1	0	3	$\frac{1}{2}(6+\sqrt{6}\lambda)$	unstable
A2.	0	0	$\frac{-3}{2}$	$\frac{3}{2}$	unstable
A3.	$\frac{\sqrt{3/2}}{\lambda}$	$\frac{\sqrt{3/2}}{\lambda}$	$\frac{3\left(-\lambda^2-\sqrt{\{24\lambda^2-7\lambda^4)}\right\}}{4\lambda^2}$	$\frac{3\left(-\lambda^2+\sqrt{\{24\lambda^2-7\lambda^4)}\right\}}{4\lambda^2}$	stable

A4.	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	$\frac{1}{2}(-6+\lambda^2)$	$-3+\lambda^2$	stable

The point A1 is an unstable because all the eigenvalues are positive. The point A2 is an unstable because one of the eigenvalue is positive. The critical point A3 is stable point because the eigenvalues are negative for λ≥1.8. The critical point A4 is stable point because the eigenvalues are negative for 0≤λ≤1.7.

Now, let's analyze the phase space and trajectories for the steep exponential potential. We will plot the phase portrait using various parameter values of λ. We have noticed that the dynamics of the phase space can be represented different sets of values for λ.

The Fig. 1 is plotted for λ=1, with the initial ranges for x and y taken as [-1, 1] and [0, 2], respectively. The attractor point in Fig. 1 corresponds to the accelerated expansion, for which $w_{eff}=-2/3$.

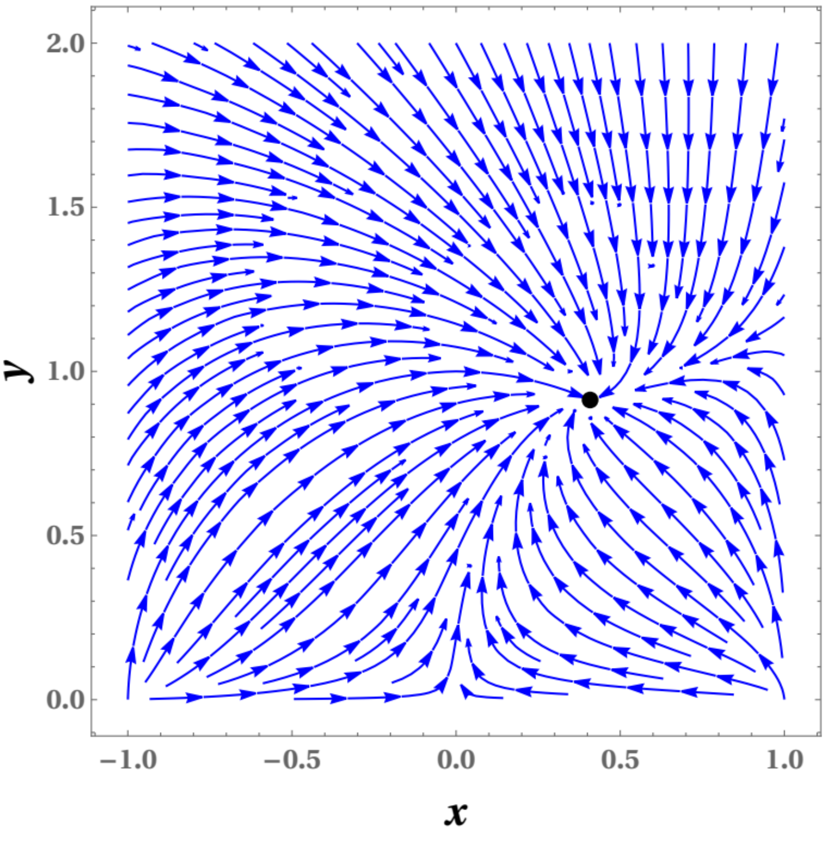

Fig. 1. This figure shows phase portrait for point A4 with λ = 1 in x-y plane. For $\omega_{eff}=-2/3$ and $\Omega\phi=1$, the point is fixed and represented a stable spiral. The stable attractor point is denoted by black dot in the figure where all the trajectories meet.

Findings and discussion

The discussion extends to cosmological models that account for the accelerating expansion of the Universe, a phenomenon frequently attributed to dark energy. Among these, models involving scalar fields – particularly quintessence – are of significant interest. For the quintessence framework, we introduce the corresponding Lagrangian density, defined by a minimally coupled scalar field with a specified potential $V(\phi)$. From this formulation, the field equations, relevant dimensionless variables, and the associated equation of state are derived. In our analysis, we focus on a steep exponential potential. We identify the fixed points of the resulting autonomous system and examine their stability by analyzing the signs of the eigenvalues. Depending on these signs, each critical point can be classified as unstable, a saddle point, or stable. The dynamics of the system are further illustrated through phase portraits. The phase-space analysis for the single exponential potential demonstrates that the system admits both stable and unstable critical points, depending on the values of the parameter λ. In particular, for certain ranges of λ, the system evolves toward a stable attractor solution, indicating long-term stability in the cosmological evolution. For parameter choices such that $\lambda<2$, the stable critical point manifests as a stable spiral, with $x \to 0$ and $y \to 1$; see Figure (1).

Conclusion

The model under consideration examines the evolution of the Universe following the Big Bang, with a focus on scalar field cosmology and dynamical system analysis. It outlines the rapid expansion of the early Universe, which eventually led to the formation of galaxies and other cosmic structures. The study also discusses the concept of dark energy, the driving force behind the accelerated expansion of the Universe, and reviews the discovery of cosmic acceleration in the late 1990s. This work explores the mathematical framework of scalar field cosmology, including dynamical systems techniques and cosmological models such as quintessence. Stationary (critical) points and their stability properties are analyzed, and phase-space diagrams are constructed for various scenarios. Through these analyses, the study evaluates several potential models, investigates their stability, and interprets their phase-space trajectories, thereby offering a comprehensive overview of scalar-field-driven cosmological evolution. By employing a dynamical systems approach, we have characterized the qualitative behavior of quintessence models. We identified their critical points, examined the corresponding phase spaces, and assessed the physical plausibility of the resulting dynamics. Our findings indicate that both forms of exponential potentials considered in this work can give rise to stable scaling solutions as well as sustained late-time accelerated expansion. In the accelerating regime, the scalar field's potential energy dominates the evolution of the Universe. In scaling solutions, the scalar field's equation of state behaves similarly to that of the background matter, causing the Universe to evolve as though it were matter-dominated. These results are particularly significant, as they show that the Universe can follow realistic evolutionary paths in the presence of a scalar field without conflicting with observational constraints. For the steep exponential potential, the phase-space analysis consistently reveals stable critical points, highlighting the viability of such models in describing late-time cosmic dynamics.

Acknowledgments

MK and MS acknowledge Integral University, Lucknow for financial support through Integral Research Fellowship (IRF) and Seed Money Grant 2024-2025 (Project Sanction No.: IUL/ICEIR/SMP/2024-04), respectively.

Conflict of interests

There is no conflict of interests.

The contribution of the authors

Muskan Khan – Performed the analysis and wrote the manuscript.

Mohd Shahalam – Developed the research idea and supervised the project.

References

1. N. Aghanim et al. (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. 641, A6 (2020), DOI: https://doi.org/10.1051/0004-6361/201833910

2. S. Perlmutter et al., Measurements of Ω and Λ from 42 High-Redshift Supernovae, Astrophys. J. 517, 565 (1999), DOI: https://ui.adsabs.harvard.edu/link_gateway/1999ApJ...517..565P/doi:10.1086/307221

3. A. G. Riess et al. (Supernova Search Team Collaboration), Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J. 116, 1009 (1998), DOI: https://ui.adsabs.harvard.edu/link_gateway/1998AJ....116.1009R/doi:10.1086/300499

4. D. N. Spergel et al., First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters, Astrophys. J. Suppl. 148, 175 (2003),. DOI: https://ui.adsabs.harvard.edu/link_gateway/2003ApJS..148..175S/doi:10.1086/377226

5. E. Komatsu et al., Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation, Astrophys. J. Suppl. 192, 18 (2011), DOI: https://ui.adsabs.harvard.edu/link_gateway/2011ApJS..192...18K/doi:10.1088/0067-0049/192/2/18

6. E. J. Copeland, M. Sami & S. Tsujikawa, Dynamics of dark energy, Int. J. Mod. Phys. D 15, 1753 (2006), arXiv:hep-th/0603057, DOI: https://doi.org/10.1142/S021827180600942X

7. V. Sahni & A. Starobinsky, The Case for a Positive Cosmological Λ-term, Int. J. Mod. Phys. D 9, 373 (2000), DOI: https://doi.org/10.1142/S0218271800000542

8. G. Hinshaw et al. (WMAP Collaboration), Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, arXiv:1212.5226 [astro-ph.CO] (2012), DOI: https://doi.org/10.1088/0067-0049/208/2/19

9. L. M. Krauss & B. Chaboyer, Age Estimates of Globular Clusters in the Milky Way: Constraints on Cosmology, Science 299, 65 (2003), DOI: https://ui.adsabs.harvard.edu/link_gateway/2003Sci...299...65K/doi:10.1126/science.1075631

10. J. Frieman, M. Turner & D. Huterer, Dark energy and the accelerating universe, Annu. Rev. Astron. Astrophys. 46, 385 (2008), DOI: https://doi.org/10.1146/annurev.astro.46.060407.145243

11. E. J. Copeland, A. R. Liddle & D. Wands, Exponential potentials and cosmological scaling solutions, Phys. Rev. D 57, 4686 (1998), arXiv:gr-qc/9711068, DOI: https://doi.org/10.1103/PhysRevD.57.4686

12. I. Percival & D. Richards, Introduction to Dynamics (Cambridge University Press, Cambridge, 1999).

13. B. Gumjudpai, T. Naskar, M. Sami & S. Tsujikawa, Coupled dark energy: towards a general description of the dynamics, JCAP 0506, 007 (2005), DOI: https://doi.org/10.1088/1475-7516/2005/06/007

14. W. Yang, M. Shahalam, B. Pal, S. Pan & A. Wang, Constraints on quintessence scalar field models using cosmological observations, Phys. Rev. D 100, 023522 (2019), DOI: https://doi.org/10.1103/PhysRevD.100.023522

15. M. Shahalam, S. Ayoub, P. Avlani & R. Myrzakulov, Dynamical system analysis in descending dark energy model, IJGMMP, 2550227 (2025) arXiv:2402.01270, DOI: https://doi.org/10.1142/S0219887825502275

Мускан Хан, Мохд Шахалам

Физика кафедрасы, Integral University, Лакнау, 226026, Үндістан

*(E-mail: *mohdshahamu@gmail.com, mk04567432@gmail.com)*

Тұрақты нүкте талдауы және тік экспоненциал потенциалы

Аңдатпа. Бұл жұмыста біз тік экспоненциалды потенциалмен басқарылатын космологиялық модельдің динамикалық әрекетін зерттейміз. Біздің зерттеуіміз модельдің асимптотикалық сипаттамаларын зерттеуге мүмкіндік беретін эволюция теңдеулерінің автономды түріне шоғырланған. Біз скаляр өрісімен және ғаламның кеңеюімен байланысты бірінші ретті дифференциалдық теңдеулер жинағын құрастырамыз. Талдаудың негізгі бөлігі космологиялық модельдің маңызды нүктелерін анықтауды қамтиды. Бұл нүктелердің тұрақтылық қасиеттерін меншікті мәндер арқылы анықтаймыз, оларды тұрақты, тұрақсыз немесе седла нүктелеріне жіктейміз. Бұл жіктеу ғаламның тік экспоненциалды әлеуеттің әсерінен қалай сапалы түрде дамуы мүмкін екендігі туралы құнды түсінік береді. Нәтижелеріміз нақты параметр диапазондары үшін үлгі кешіктірілген аттрактор ретінде жұмыс істейтін тұрақты сыни нүктені қабылдайтынын көрсетеді. Бұл экспоненциалды потенциал ғаламды эволюцияның тұрақты фазасына апара алатынын көрсетеді. Сонымен қатар, біз ағымдағы дәуірдегі скалярлық өріс үстемдігінің нақты визуализациясын ұсына отырып, жүйенің траекториялары мен тұрақтылық құрылымын суреттеу үшін фазалық кеңістік диаграммаларын қолданамыз.

Түйін сөздер: Динамикалық талдау, Тұрақты нүкте, Квинтессенция, Ғарыштық үдеу, Фазалық кеңістік

Мускан Хан, Мохд Шахалам

Кафедра физики, Интегральный университет, Лукнау, 226026, Индия

*(E-mail: *mohdshahamu@gmail.com, mk04567432@gmail.com)*

Анализ стационарных точек для крутого экспоненциального потенциала

Аннотация. В данной работе мы исследуем динамическое поведение космологической модели, управляемой крутым экспоненциальным потенциалом. Наше исследование сосредоточено на автономной форме уравнений эволюции, что позволяет нам исследовать асимптотические характеристики модели. Мы строим систему дифференциальных уравнений первого порядка, связанную со скалярным полем и расширением Вселенной. Ключевая часть анализа заключается в выявлении критических точек космологической модели. Мы определяем свойства устойчивости этих точек через собственные значения, классифицируя их как устойчивые, неустойчивые или седловые. Эта классификация даёт ценную информацию о том, как Вселенная может качественно эволюционировать под воздействием крутого экспоненциального потенциала. Наши результаты показывают, что для определённых диапазонов параметров модель допускает устойчивую критическую точку, которая действует как аттрактор позднего времени. Это указывает на то, что экспоненциальный потенциал может подталкивать Вселенную к стабильной фазе эволюции. Кроме того, мы используем диаграммы фазового пространства для иллюстрации траекторий и структуры устойчивости системы, предлагая наглядную визуализацию доминирования скалярного поля в текущую эпоху.

Ключевые слова: Динамический анализ, Неподвижная точка, Квинтэссенция, Космическое ускорение, Фазовое пространство

References

1. N. Aghanim et al. (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. 641, A6 (2020), DOI: https://doi.org/10.1051/0004-6361/201833910

2. S. Perlmutter et al., Measurements of Ω and Λ from 42 High-Redshift Supernovae, Astrophys. J. 517, 565 (1999), DOI: https://ui.adsabs.harvard.edu/link_gateway/1999ApJ...517..565P/doi:10.1086/307221

3. A. G. Riess et al. (Supernova Search Team Collaboration), Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J. 116, 1009 (1998), DOI: https://ui.adsabs.harvard.edu/link_gateway/1998AJ....116.1009R/doi:10.1086/300499

4. D. N. Spergel et al., First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters, Astrophys. J. Suppl. 148, 175 (2003),. DOI: https://ui.adsabs.harvard.edu/link_gateway/2003ApJS..148..175S/doi:10.1086/377226

5. E. Komatsu et al., Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation, Astrophys. J. Suppl. 192, 18 (2011), DOI: https://ui.adsabs.harvard.edu/link_gateway/2011ApJS..192...18K/doi:10.1088/0067-0049/192/2/18

6. E. J. Copeland, M. Sami & S. Tsujikawa, Dynamics of dark energy, Int. J. Mod. Phys. D 15, 1753 (2006), arXiv:hep-th/0603057, DOI: https://doi.org/10.1142/S021827180600942X

7. V. Sahni & A. Starobinsky, The Case for a Positive Cosmological Λ-term, Int. J. Mod. Phys. D 9, 373 (2000), DOI: https://doi.org/10.1142/S0218271800000542

8. G. Hinshaw et al. (WMAP Collaboration), Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, arXiv:1212.5226 [astro-ph.CO] (2012), DOI: https://doi.org/10.1088/0067-0049/208/2/19

9. L. M. Krauss & B. Chaboyer, Age Estimates of Globular Clusters in the Milky Way: Constraints on Cosmology, Science 299, 65 (2003), DOI: https://ui.adsabs.harvard.edu/link_gateway/2003Sci...299...65K/doi:10.1126/science.1075631

10. J. Frieman, M. Turner & D. Huterer, Dark energy and the accelerating universe, Annu. Rev. Astron. Astrophys. 46, 385 (2008), DOI: https://doi.org/10.1146/annurev.astro.46.060407.145243

11. E. J. Copeland, A. R. Liddle & D. Wands, Exponential potentials and cosmological scaling solutions, Phys. Rev. D 57, 4686 (1998), arXiv:gr-qc/9711068, DOI: https://doi.org/10.1103/PhysRevD.57.4686

12. I. Percival & D. Richards, Introduction to Dynamics (Cambridge University Press, Cambridge, 1999).

13. B. Gumjudpai, T. Naskar, M. Sami & S. Tsujikawa, Coupled dark energy: towards a general description of the dynamics, JCAP 0506, 007 (2005), DOI: https://doi.org/10.1088/1475-7516/2005/06/007

14. W. Yang, M. Shahalam, B. Pal, S. Pan & A. Wang, Constraints on quintessence scalar field models using cosmological observations, Phys. Rev. D 100, 023522 (2019), DOI: https://doi.org/10.1103/PhysRevD.100.023522

15. M. Shahalam, S. Ayoub, P. Avlani & R. Myrzakulov, Dynamical system analysis in descending dark energy model, IJGMMP, 2550227 (2025) arXiv:2402.01270, DOI: https://doi.org/10.1142/S0219887825502275

Information about the authors

Muskan Khan **–** PhD student, Department of Physics, Integral University, Lucknow 226026, India

Mohd Shahalam **–** the corresponding author, PhD, Professor of Department of Physics, Integral University, Lucknow 226026, India

Мұсқан Хан **–** PhD студенті, Интеграл университетінің физика факультеті, Лакхнау 226026, Үндістан

Мохд Шахалам **–** хат-хабар авторы, PhD, Интеграл университетінің физика кафедрасының профессоры, Лакхнау 226026, Үндістан

Маскан Хан **–** PhD студент, факультет физики, Интегральный университет, Лакхнау 226026, Индия

Мохд Шахалам **–** автор для корреспонденции, PhD, профессор кафедры физики Интегрального университета, Лакхнау 226026, Индия